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Research article

Picture fuzzy multifunctions and modal topological structures

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Abstract: This paper introduces the notion of picture fuzzy modal topological multifunctions. These structures are grounded on novel picture fuzzy topological operators for closure and interior types, thereby utilizing the two standard picture fuzzy modal operators $\Box$ and $\Diamond$. The paper discusses several fundamental properties of picture fuzzy multifunctions. Many types of structures are introduced, and some types of continuous multifunctions between picture fuzzy topological structures are discussed. The results indicate that some properties considered satisfactory in the intuitionistic fuzzy modal topological structures, as defined by Atanassov in 2022, are not fulfilled.

Keywords: picture fuzzy multifunction; picture fuzzy modal topology; picture fuzzy operator
Mathematics Subject Classification: 03B20, 03B52, 03E72, 03E75, 94D05

Abbreviations

These abbreviations are used in this manuscript.

FS $\Rightarrow$ fuzzy set, IFS $\Rightarrow$ intuitionistic fuzzy set, PFS $\Rightarrow$ picture fuzzy set,
PFMTS $\Rightarrow$ picture fuzzy modal topological structure,
PFFMTS $\Rightarrow$ picture fuzzy feeble modal topological structure,
PFTM $\Rightarrow$ picture fuzzy topological multifunction,
PF^uS $\Rightarrow$ picture fuzzy upper semi, PF^lS $\Rightarrow$ picture fuzzy lower semi,
PF^uA $\Rightarrow$ picture fuzzy upper almost, PF^lA $\Rightarrow$ picture fuzzy lower almost,
PF^uW $\Rightarrow$ picture fuzzy upper weakly, PF^lW $\Rightarrow$ picture fuzzy lower weakly,
PF^uAW $\Rightarrow$ picture fuzzy upper almost weakly,
PF^lAW $\Rightarrow$ picture fuzzy lower almost weakly.

1. Introduction

Fuzzification is a crucial tool for addressing humanistic systems in real-life problems. The seminal paper on the fuzzy set (FS) theory was authored by Zadeh in 1965 ([28]). This theory of FSs has been widely applied by many scholars. The fuzzy set theory described the positivism of an element ξ of a universal set ξ to a subset $\mathbb{K} \subseteq \Xi$ by the membership value $\omega_{\mathbb{K}}(\xi)$, and posited that the negativism of that element $\xi \in \Xi$ to the set $\mathbb{K}$ is $1 - \omega_{\mathbb{K}}(\xi)$. In [7], Atanassov based his theory of intuitionistic fuzzy sets (IFSs) on the notion that the negativism $\varpi_{\mathbb{K}}(\xi)$ of an element $\xi \in \Xi$ to a subset $\mathbb{K} \subseteq \Xi$ may range from $[0, 1]$ and does not need to be the complement of the positivism of that element $\xi \in \Xi$ to $\mathbb{K}$. The values $\omega_{\mathbb{K}}(\xi)$ and $\varpi_{\mathbb{K}}(\xi)$ represent the positivism and negativism of each $\xi \in \Xi$ to $\mathbb{K}$, respectively, with the condition that $0 \leq \omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) \leq 1$. In this way, Atanassov encompassed all the FSs as a special case of his theory whenever $\omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) = 1$. IFSs are more meaningful and applicable to real-life problems. In [12], Cuong introduced the theory of picture fuzzy sets (PFSs) by adding the neutralism of an element $\xi \in \Xi$ to the subset $\mathbb{K}$, represented by $\sigma_{\mathbb{K}}(\xi)$. This definition is conditioned with $0 \leq \omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi) \leq 1$. Then, in the case where $\sigma_{\mathbb{K}}(\xi) = 0$ for all $\xi \in \Xi$, we revert to $\mathbb{K}$ in the IFS. Moreover, if $\varpi_{\mathbb{K}}(\xi) = 1 - \omega_{\mathbb{K}}(\xi)$, then we revert to $\mathbb{K}$ in the FS. There are several simple modifications for the IFSs [9, 27], which we shall not discuss here. These modifications include Pythagorean FSs [20, 23], Fermatean FSs [21], Spherical FSs [6, 16], q-rung orthopair FSs [4, 24], q-rung orthopair PFSs [17], $(\varsigma, \varkappa)$-fuzzy local functions, continuous multifunctions, and double fuzzy ideal topological spaces [2, 3]. All these definitions, starting from FSs, have applications in image processing, decision theory, uncertainty modeling, and beyond, as in [10, 14, 26].

In this paper, we merge the classical definitions of multifunctions in general topology and the standard modal logic [13, 19] with the notion of PFSs, thus further expanding into the realm of picture fuzzy modal topological structures (PFMTSs). This exploration includes the creation of PFMTSs facilitated by the standard picture fuzzy operations of "union" ($\cup$) and "intersection" ($\cap$). Continuous functions between picture fuzzy topological spaces were discussed in [1].

The motivations of this paper are as follow: Section 2 presents fuzzy modal topological structures related to the picture fuzzy sets, and studying some important results including several modal operators; (2) picture fuzzy topological multifunctions and their common results. These results are given in Section 3; (3) some types of continuity of picture fuzzy multifunctions, and study the possible implications; and finally, the conclusions are given in Section 5.

The research on PFMTSs has several important applications in domains: decision making, pattern recognition, artificial intelligence, information retrieval and data mining. PFMTSs address critical gaps in handling uncertainty, imprecision, and neutrality, which are inherent in real-life problems across diverse domains. To bridge these gaps, PFSs were introduced, adding a neutrality component to the membership and non-membership values, thereby enabling a more nuanced representation of uncertainty. PFMTSs expand upon these concepts by the integration in modal logic and general topology using the PFSs. This integration introduces global operators, such as closure, interior, and modal operators ($\Box$ and $\Diamond$), which modify classical topological and modal relationships. These global operators facilitate a robust analysis of FSs under modal and topological constraints, providing a suitable tools for theoretical exploration and practical applications. The study of PFMTSs not only extends the theory of FSs but also establishes a wide platform to address modern computational challenges. Its ability to integrate neutrality, positivity, and negativity within a unified framework lays the foundation for the further exploration and application of PFMTSs in dynamic systems, hybrid models, and emerging technologies, thus positioning it as a cornerstone of modern mathematical and computational innovation.

PFMTSs have special important applications in the Artificial Intelligent work.

Fuzzy Logic Systems: PFMTSs can enhance fuzzy logic systems, which are used to handle uncertain or imprecise information, thus improving decision making and reasoning processes. Additionally, it can improve the expert systems by incorporating various degrees of truth, belief, or doubt, thus enabling more braced and human liker decision making [5, 15, 22].

Natural Language Process: PFMTSs can be used in the natural language process for semantic analyses, thus allowing systems to understand and process Language with varying degrees of ambiguity or multiple interpretations [11, 18].

Machine Learning: The research can benefit machine learning models by enhancing clustering and classification algorithms, particularly when dealing with ambiguous data. PFMTSs can be applied to decision-making and control systems, helping robots to navigate and interact in uncertain or dynamic systems [25].

2. Picture fuzzy modal topological structures

Continuing from previous discussions and the notions given by Atanassov in [8,9], let's define a PFS $\mathbb{K}$ on the universal set ξ. The set $\mathbb{K}$ consists of elements $\xi \in \Xi$, each described by degrees of positivism ($\omega_{\mathbb{K}}(\xi)$), negativism ($\varpi_{\mathbb{K}}(\xi)$), and neutralism ($\sigma_{\mathbb{K}}(\xi)$) that lie within the interval $[0, 1]$. Specifically, $\mathbb{K}$ is represented as $\{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$, where each component satisfies the condition $0 \leq \omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi) \leq 1$ for every element x. The term $\pi_{\mathbb{K}}(\xi) = 1 - (\omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi))$ indicates the degree of the refusal membership value for each ξ in $\mathbb{K}$, quantifying the extent to which ξ does not belong to $\mathbb{K}$. This framework is pivotal to assess and handle the nuances of membership within PFSs, enabling a more comprehensive analysis of elements based on their multiple affinities.

Definition 2.1. *[4] Let ξ be a nonempty set, $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$ and $\text{Ч} = \{\langle \xi, \omega_{\text{Ч}}(\xi), \varpi_{\text{Ч}}(\xi), \sigma_{\text{Ч}}(\xi)\rangle \,|\xi \in \Xi\}$. Then,*

(1) $\mathbb{K} \subseteq \text{Ч}$ *iff for all* $\xi \in \Xi$, $\omega_{\mathbb{K}}(\xi) \leq \omega_{\text{Ч}}(\xi)$, $\varpi_{\mathbb{K}}(\xi) \geq \varpi_{\text{Ч}}(\xi)$ *and* $\sigma_{\mathbb{K}}(\xi) \leq \sigma_{\text{Ч}}(\xi)$ *or* $\sigma_{\mathbb{K}}(\xi) \geq \sigma_{\text{Ч}}(\xi)$*;*

(2) $\mathbb{K} = \text{Ч}$ *iff for all* $\xi \in \Xi$, $\omega_{\mathbb{K}}(\xi) = \omega_{\text{Ч}}(\xi)$, $\varpi_{\mathbb{K}}(\xi) = \varpi_{\text{Ч}}(\xi)$ *and* $\sigma_{\mathbb{K}}(\xi) = \sigma_{\text{Ч}}(\xi)$*;*

(3) $\mathbb{K} \cup \text{Ч} = \{\left\langle \begin{array}{c} \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{\text{Ч}}(\xi)), (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{\text{Ч}}(\xi)), \\ (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)) \end{array} \right\rangle |\xi \in \Xi\}$;

(4) $\mathbb{K} \cap \text{Ч} = \{\left\langle \begin{array}{c} \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}(\xi)), \\ (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)) \end{array} \right\rangle |\xi \in \Xi\}$;

(5) $\mathbb{K} \Cup \text{Ч} = \{\left\langle \begin{array}{c} \xi, 1 - (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{\text{Ч}}(\xi)) - (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)), \\ (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{\text{Ч}}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)) \end{array} \right\rangle |\xi \in \Xi\}$;

(6) $\mathbb{K} \Cap \text{Ч} = \{\left\langle \begin{array}{c} \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)), \\ 1 - (\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)) - (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)) \end{array} \right\rangle |\xi \in \Xi\}$;

(7) $\dashv\mathbb{K} = \{\langle \xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$.

Now, the definitions of standard two modal operators over (PFS) are presented:

$$\Box\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\},$$

$$\Diamond\mathbb{K} = \{\langle \xi, 1-\varpi_{\mathbb{K}}(\xi)-\sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}.$$

We can see that $\Box\mathbb{K} \subseteq \mathbb{K} \subseteq \Diamond\mathbb{K}$ in general, and $\Box\mathbb{K} \neq \mathbb{K} \neq \Diamond\mathbb{K}$ for any proper set $\mathbb{K}$ in (PFS), that is, $\omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi) < 1$ with $\sigma_{\mathbb{K}}(\xi) \neq 0$ for every element ξ. Otherwise, $\mathbb{K}$ is an IFS and still $\Box\mathbb{K} \subseteq \mathbb{K} \subseteq \Diamond\mathbb{K}$ as usual in (IFS). Moreover, if $\mathbb{K}$ is a non proper PFS and $\varpi_{\mathbb{K}}(\xi) = 1-\omega_{\mathbb{K}}(\xi)$, then $\mathbb{K}$ is a FS in (FS) and $\Box\mathbb{K} = \mathbb{K} = \Diamond\mathbb{K}$. Thus, $(FS) \subseteq (IFS) \subseteq (PFS)$.

$$\text{Let } \flat = \{\langle \xi, 0, 1, 0\rangle \,|\xi \in \Xi\}, \quad \natural = \{\langle \xi, 0, 0, 1\rangle \,|\xi \in \Xi\}, \quad \sharp = \{\langle \xi, 1, 0, 0\rangle \,|\xi \in \Xi\},$$

where $\flat \subseteq \mathbb{K} \subseteq \sharp$ for all $\mathbb{K} \in (PFS)$. Normally, $\mathcal{P}(\flat) = \{\flat\}$ and $\mathcal{P}(\sharp) = \{\mathbb{K}|\mathbb{K} \subseteq \sharp\}$ where $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$; therefore, (PFS) coincides with $\mathcal{P}(\sharp)$. (IFS) coincides with the set $\{\mathbb{K}|\mathbb{K} \subseteq \Xi\}$ in which $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), 0\rangle \,|\xi \in \Xi\}$. Moreover, (FS) coincides with the set $\{\mathbb{K}|\mathbb{K} \subseteq \Xi\}$ in which $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1-\omega_{\mathbb{K}}(\xi), 0\rangle \,|\xi \in \Xi\}$ or $\mathbb{K} = \{\langle \xi, 1-\varpi_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), 0\rangle \,|\xi \in \Xi\}$.

Based on the notions in [8], we will call a cl-PFMT by the object $\langle \mathcal{P}(\sharp), cl, \triangle, \triangledown, \circ\rangle$ where Ξ is a fixed universe, $cl : \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ is an operator over Ξ, $\triangle, \triangledown : \mathcal{P}(\sharp) \times \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ are operations over Ξ such that for every two $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$, we have $\mathbb{K}\triangledown$ Ч $= \dashv(\dashv\mathbb{K}\triangle\dashv$ Ч$)$.

$\circ : \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ is a modal operator over X, and for every two $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$.

C1 $cl(\mathbb{K}\triangle$ Ч$) = cl(\mathbb{K})\triangle\, cl($Ч$)$,

C2 $\mathbb{K} \subseteq cl(\mathbb{K})$,

C3 $cl(\flat) = \flat$,

C4 $cl(cl(\mathbb{K})) = cl(\mathbb{K})$,

C5 $\circ(\mathbb{K}\triangledown$ Ч$) = \circ(\mathbb{K})\triangledown \circ($Ч$)$,

C6 $\circ(\mathbb{K}) \subseteq \mathbb{K}$,

C7 $\circ(\sharp) = \sharp$,

C8 $\circ(\circ(\mathbb{K})) = \circ(\mathbb{K})$,

C9 $\circ(cl(\mathbb{K})) = cl(\circ(\mathbb{K}))$.

Note: Not all conditions are applicable to every element within PFMTSs. In certain instances, some conditions are absent, and in others, the relationships defined by these conditions are characterized by weak (feeble) connections. For such structures, the term "feeble" is employed, and these are referred to as picture fuzzy feeble modal topological structures (PFFMTSs). For these PFFMTSs, we introduce analogs to the topological operators "closure" and "interior" for PFSs as follows: $cl_{\cap}(\mathbb{K}) = \{\langle \xi, \epsilon_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle|\xi \in \Xi\}$, $int_{\cap}(\mathbb{K}) = \{\langle \xi, \varepsilon_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \varkappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\}$, $cl_{\Cup}(\mathbb{K}) = \{\langle \xi, 1-\aleph_{\mathbb{K}}-\kappa_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\}$ and $int_{\Cap}(\mathbb{K}) = \{\langle \xi, \varepsilon_{\mathbb{K}}, 1-\varepsilon_{\mathbb{K}}-\kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\}$, where

$$\begin{aligned}
\epsilon_{\mathbb{K}} &= \bigvee_{\xi\in\Xi} \omega_{\mathbb{K}}(\xi), \quad \aleph_{\mathbb{K}} = \bigwedge_{\xi\in\Xi} \varpi_{\mathbb{K}}(\xi), \quad \kappa_{\mathbb{K}} = \bigwedge_{\xi\in\Xi} \sigma_{\mathbb{K}}(\xi) \\
\varepsilon_{\mathbb{K}} &= \bigwedge_{\xi\in\Xi} \omega_{\mathbb{K}}(\xi), \quad \vartheta_{\mathbb{K}} = \bigvee_{\xi\in\Xi} \varpi_{\mathbb{K}}(\xi), \quad \varkappa_{\mathbb{K}} = \bigvee_{\xi\in\Xi} \sigma_{\mathbb{K}}(\xi).
\end{aligned}$$

Therefore,

(1) $\dashv cl_{\Cup}(\dashv\mathbb{K}) = int_{\Cap}(\mathbb{K})$,

(2) $\dashv int_{\Cap}(\dashv\mathbb{K}) = cl_{\Cup}(\mathbb{K})$,

(3) $int_{\Cap}(\mathbb{K}) \subseteq int_{\cap}(\mathbb{K}) \subseteq cl_{\cap}(\mathbb{K}) \subseteq cl_{\Cup}(\mathbb{K})$.

These definitions expand the conceptual framework of topological operations within the context of PFSs, thus accommodating the variability and flexibility required by the feeble relationships in

PFFMTSs. Always, for any $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$, the De Morgan's laws are satisfied, that is,

$$\mathbb{K} \cup Ч = \exists(\exists\mathbb{K} \cap \exists Ч) \text{ and } \mathbb{K} \cap Ч = \exists(\exists\mathbb{K} \cup \exists Ч).$$

Theorem 2.2. $\langle \mathcal{P}(\sharp), \mathrm{cl}_{\cap}, \cup, \cap, \square \rangle$ *is a* cl- *PFFMT for which the equality relation* " = " *is changed to the inclusion relation* " $\subseteq$ " *in conditions (C5) and (C9).*

Proof. Let $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$. Then, we will check the validity of all the nine conditions C1–C9.
$(C1)$

$$\begin{aligned} cl_{\cap}(\mathbb{K} \cup Ч) =& cl_{\cap}(\{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\} \cup \{\langle \xi, \omega_{Ч}(\xi), \varpi_{Ч}(\xi), \sigma_{Ч}(\xi)\rangle | \xi \in \Xi\}) \\ =& cl_{\cap}(\{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\rangle | \xi \in \Xi\}) \\ =& \{\left\langle \xi, \bigvee_{\Upsilon \in \xi} (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), \bigwedge_{\Upsilon \in \xi} (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{Ч}(\xi)), \bigwedge_{\Upsilon \in \xi} (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\right\rangle | \xi \in \Xi\} \\ =& \{\langle \xi, (\epsilon_{\mathbb{K}} \vee \epsilon_{Ч}), (\aleph_{\mathbb{K}} \wedge \aleph_{Ч}), (\kappa_{\mathbb{K}} \wedge \kappa_{Ч})\rangle | \xi \in \Xi\} \\ =& cl_{\cap}(\mathbb{K}) \cup cl_{\cap}(Ч), \end{aligned}$$

$(C2)$

$$\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\} \subseteq \{\langle \xi, \epsilon_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle | \xi \in \Xi\} = cl_{\cap}(\mathbb{K}),$$

$(C3)$

$$cl_{\cap}(\flat) = cl_{\cap}(\{\langle \xi, 0, 1, 0\rangle | \xi \in \Xi\}) = \{\left\langle \xi, \bigvee_{\xi \in \Xi} 0, \bigwedge_{\xi \in \Xi} 1, \bigwedge_{\xi \in \Xi} 0\right\rangle | \xi \in \Xi\} = \{\langle \xi, 0, 1, 0\rangle | \xi \in \Xi\} = \flat,$$

$(C4)$

$$cl_{\cap}(cl_{\cap}(\mathbb{K})) = cl_{\cap}(\{\langle \xi, \epsilon_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle | \xi \in \Xi\}) = \{\langle \xi, \epsilon_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle | \xi \in \Xi\} = cl_{\cap}(\mathbb{K}),$$

$(C5)$

$$\begin{aligned} &\square(\mathbb{K} \cap Ч) \\ =& \square(\{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\rangle | \xi \in \Xi\}) \\ =& \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)), 1 - (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)) - (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\rangle | \xi \in \Xi\} \\ \subseteq& \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)), ((1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi)) \vee (1 - \omega_{Ч}(\xi) - \sigma_{Ч}(\xi))), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\rangle | \xi \in \Xi\} \\ =& \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\} \cap \{\langle \xi, \omega_{Ч}(\xi), 1 - \omega_{Ч}(\xi) - \sigma_{Ч}(\xi), \sigma_{Ч}(\xi)\rangle | \xi \in \Xi\} \\ =& \square(\mathbb{K}) \cap \square(Ч), \end{aligned}$$

$(C6)$

$$\square(\mathbb{K}) = \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\} \subseteq \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\Xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\} = \mathbb{K},$$

$(C7)$

$$\square(\sharp) = \square(\{\langle \xi, 1, 0, 0\rangle | \xi \in \Xi\}) = \{\langle \xi, 1, 0, 0\rangle | \xi \in \Xi\} = \sharp,$$

$(C8)$

$$\begin{aligned} \square(\square(\mathbb{K})) =& \square(\{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\}) \\ =& \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle | \xi \in \Xi\} = \square(\mathbb{K}), \end{aligned}$$

($C9$)

$$\Box(cl_{\cap}(\mathbb{K})) = \Box(\{\langle \xi, \epsilon_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}} \rangle \,|\xi \in \Xi\}) = \{\langle \xi, \epsilon_{\mathbb{K}}, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}} \rangle \,|\xi \in \Xi\}$$
$$= \{\left\langle \xi, \epsilon_{\mathbb{K}}, \bigwedge_{\xi\in\Xi}(1 - \omega_{\mathbb{K}}(\xi)) - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}} \right\rangle |\xi \in \Xi\}.$$

□

Example 2.3. *Let* $\Xi = \{\xi_1, \xi_2\}$. *Then,*

(1) *For* $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$, $\mathbb{K} = \{\langle \xi, 0.3, 0.5, 0.1\rangle \,|\xi \in \Xi\}$ *and* Ч $= \{\langle \xi, 0.2, 0.6, 0.2\rangle \,|\xi \in \Xi\}$,
$\Box\mathbb{K} = \{\langle \xi, 0.3, 0.6, 0.1\rangle |\xi \in \Xi\}$, $\Box$ Ч $= \{\langle \xi, 0.2, 0.6, 0.2\rangle \,|\xi \in \Xi\}$,
$\Box(\mathbb{K}) \cap \Box($Ч$) = \{\langle \xi, 0.2, 0.6, 0.1\rangle \,|\xi \in \Xi\}$, $\mathbb{K} \cap$ Ч $= \{\langle \xi, 0.2, 0.6, 0.1\rangle \,|\xi \in \Xi\}$,
$\Box(\mathbb{K} \cap$ Ч$) = \{\langle \xi, 0.2, 0.7, 0.1\rangle \,|\xi \in \Xi\} \neq \Box(\mathbb{K}) \cap \Box($Ч$)$.

(2) *For* $\mathbb{K} \in \mathcal{P}(\sharp)$, $\mathbb{K} = \{\langle \xi_1, 0.2, 0.1, 0.1\rangle$, $\langle \xi_2, 0.6, 0.2, 0.2\rangle \,|\xi_1, \xi_2 \in \Xi\}$,
$\mathrm{cl}_{\cap}(\mathbb{K}) = \{\langle \xi, 0.6, 0.1, 0.1\rangle \,|\xi \in \Xi\}$, $\Box(\mathrm{cl}_{\cap}(\mathbb{K})) = \{\langle \xi, 0.6, 0.3, 0.1\rangle \,|\xi \in \Xi\}$,
$\Box(\mathbb{K}) = \{\langle \xi_1, 0.2, 0.7, 0.1\rangle$, $\langle \xi_2, 0.6, 0.2, 0.2\rangle \,|\xi_1, \xi_2 \in \Xi\}$,
$\mathrm{cl}_{\cap}(\Box(\mathbb{K})) = \{\langle \xi, 0.6, 0.2, 0.1\rangle \,|\xi \in \Xi\} \neq \Box(\mathrm{cl}_{\cap}(\mathbb{K}))$.

Note: If we take $\sigma_{\mathbb{K}}(\xi) = \sigma_{Ч}(\xi) = 0$, then the relations in C5 and C9 are " = " (see Atanassov [8]).

Now, we can prove for each $\mathbb{K} \in \mathcal{P}(\sharp)$) that: $int_{\cap}(\mathbb{K}) = \dashv cl_{\cap}(\dashv\mathbb{K})$ and $\Diamond(\mathbb{K}) = \dashv\Box(\dashv\mathbb{K})$. Since

$$\begin{aligned}
\dashv\ & cl_{\cap}(\dashv\mathbb{K}) = \dashv cl_{\cap}(\{\langle \xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}) \\
=\ & \dashv(\{\langle \xi, \vartheta_{\mathbb{K}}, \varepsilon_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\}) \\
=\ & \{\langle \xi, \varepsilon_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\} = int_{\cap}(\mathbb{K}). \\
\dashv\ & \Box(\dashv\mathbb{K}) = \dashv\Box(\{\langle \xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}) \\
=\ & \dashv(\{\langle \xi, \varpi_{\mathbb{K}}(\xi), 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}) \\
=\ & \{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} = \Diamond\mathbb{K}.
\end{aligned}$$

In comparison to the above, we will call an *int*-PFMT by the object $\langle\ \mathcal{P}(\sharp), int, \triangledown, \vartriangle, *\ \rangle$ where Ξ is a fixed universe, $int : \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ is an operator over Ξ, $\triangledown, \vartriangle : \mathcal{P}(\sharp) \times \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ are operations over Ξ satisfying De Morgan's laws.

$* : \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ is a modal operator over ξ; for every $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$.

D1 $int(\mathbb{K}\triangledown$ Ч$) = int(\mathbb{K})\triangledown int($Ч$)$,
D2 $int(\mathbb{K}) \subseteq \mathbb{K}$,
D3 $int(\sharp) = \sharp$,
D4 $int(int(\mathbb{K})) = int(\mathbb{K})$,
D5 $*(\mathbb{K}\vartriangle$ Ч$) = *(mathbbK)\vartriangle *($Ч$)$,
D6 $\mathbb{K} \subseteq *(\mathbb{K})$,
D7 $*(\flat) = \flat$,
D8 $*(*(\mathbb{K})) = *(\mathbb{K})$,
D9 $*(int(\mathbb{K})) = int(*(\mathbb{K}))$.

Theorem 2.4. $\langle\ \mathcal{P}(\sharp), \mathrm{int}_{\cap}, \cap, \cup, \Diamond\ \rangle$ *is an* int-*PFFMT for which the equality relation* " = " *is changed to the inclusion relation* " $\supseteq$ " *in conditions (D5) and (D9).*

Proof. The proof is similar to the proof of Theorem 2.2. □

Example 2.5. *Let* $\Xi = \{\xi_1, \xi_2\}$. *Then,*

(1) *For* $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$, $\mathbb{K} = \{\langle \xi, 0.5, 0.3, 0.1\rangle \,|\xi \in \Xi\}$ *and* Ч $= \{\langle \xi, 0.6, 0.2, 0.2\rangle \,|\xi \in \Xi\}$,
$\diamond(\mathbb{K}) = \{\langle \xi, 0.6, 0.3, 0.1\rangle \,|\xi \in \Xi\}$, $\diamond($Ч$) = \{\langle \xi, 0.6, 0.2, 0.2\rangle \,|\xi \in \Xi\}$,
$\diamond(\mathbb{K}) \cup \diamond($Ч$) = \{\langle \xi, 0.6, 0.2, 0.1\rangle \,|\xi \in \Xi\}$, $\mathbb{K} \cup$ Ч $= \{\langle \xi, 0.6, 0.2, 0.1\rangle \,|\xi \in \Xi\}$,
$\diamond(\mathbb{K} \cup$ Ч$) = \{\langle \xi, 0.7, 0.2, 0.1\rangle \,|\xi \in \Xi\} \neq \diamond(\mathbb{K}) \cup \diamond($Ч$)$.

(2) *For* $\mathbb{K} \in \mathcal{P}(\sharp)$, $\mathbb{K} = \{\langle \xi_1, 0.1, 0.2, 0.1\rangle\ \langle \xi_2, 0.2, 0.6, 0.2\rangle \,|\xi_1, \xi_2 \in \Xi\}$,
$\mathrm{int}_\cap(\mathbb{K}) = \{\langle \xi, 0.1, 0.6, 0.1\rangle \,|\xi \in \Xi\}$, $\diamond(\mathrm{int}_\cap(\mathbb{K})) = \{\langle \xi, 0.3, 0.6, 0.1\rangle \,|\xi \in \Xi\}$,
$\diamond(\mathbb{K}) = \{\langle \xi_1, 0.7, 0.2, 0.1\rangle\ \langle \xi_2, 0.2, 0.6, 0.2\rangle \,|\xi_1, \xi_2 \in \Xi\}$,
$\mathrm{int}_\cap(\diamond(\mathbb{K})) = \{\langle \xi, 0.2, 0.6, 0.1\rangle \,|\xi \in \Xi\} \neq \diamond(\mathrm{int}_\cap(\mathbb{K}))$.

Note: If we take $\sigma_{\mathbb{K}}(\xi) = \sigma_{Ч}(\xi) = 0$, then the relations in D5 and D9 will be " = " (see Atanassov [8]).

Theorem 2.6. *For every* $\mathbb{K}$, Ч $\in \mathcal{P}(\sharp)$, *we have*

(1) $\mathrm{int}_\cap(\mathbb{K}) \subset \mathbb{K} \subset \mathrm{cl}_\cap(\mathbb{K})$,
(2) $\mathrm{cl}_\cap(\mathrm{int}_\cap(\mathbb{K})) = \mathrm{int}_\cap(\mathbb{K})$,
(3) $\mathrm{int}_\cap(\mathrm{cl}_\cap(\mathbb{K})) = \mathrm{cl}_\cap(\mathbb{K})$,
(4) $\mathrm{cl}_\cap(\mathbb{K} \cap$ Ч$) \subseteq \mathrm{cl}_\cap(\mathbb{K}) \cap \mathrm{cl}_\cap($Ч$)$,
(5) $\mathrm{int}_\cap(\mathbb{K} \cup$ Ч$) \supseteq \mathrm{int}_\cap(\mathbb{K}) \cup \mathrm{int}_\cap($Ч$)$,
(6) $\mathrm{cl}_\cap(\sharp) = \sharp$,
(7) $\mathrm{cl}_\cap(\natural) = \natural$,
(8) $\mathrm{int}_\cap(\flat) = \flat$,
(9) $\mathrm{int}_\cap(\natural) = \natural$,
(10) $\square(\mathrm{cl}_\cap(\mathbb{K})) \subseteq \mathrm{cl}_\cap(\square(\mathbb{K}))$,
(11) $\square(\mathrm{int}_\cap(\mathbb{K})) \subseteq \mathrm{int}_\cap(\square(\mathbb{K}))$,
(12) $\diamond(\mathrm{cl}_\cap(\mathbb{K})) \supseteq \mathrm{cl}_\cap(\diamond(\mathbb{K}))$,
(13) $\diamond(\mathrm{int}_\cap(\mathbb{K})) \supseteq \mathrm{int}_\cap(\diamond(\mathbb{K}))$.

Proof. We check (4) and (11).
(4)

$$\begin{aligned} cl_\cap(\mathbb{K} \cap Ч) =& cl_\cap(\{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)),\ (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \vee \sigma_{Ч}(\xi))\rangle \,|\xi \in \Xi\}) \\ =& \{\left\langle \xi, \bigvee_{\xi\in\Xi}(\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)),\ \bigwedge_{\xi\in\Xi}(\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), \bigwedge_{\xi\in\Xi}(\sigma_{\mathbb{K}}(\xi) \vee \sigma_{Ч}(\xi))\right\rangle |\xi \in \Xi\} \\ \subseteq& \{\langle \xi, (\epsilon_{\mathbb{K}} \vee \epsilon_{Ч}),\ (\aleph_{\mathbb{K}} \vee \aleph_{Ч}), (\kappa_{\mathbb{K}} \vee \kappa_{Ч})\rangle \,|\xi \in \Xi\} \\ =& \{\langle \xi, \epsilon_{\mathbb{K}},\ \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\} \cap \{\langle \xi, \epsilon_{Ч},\ \aleph_{Ч}, \kappa_{Ч}\ \rangle \,|\xi \in \Xi\} = cl_\cap(\mathbb{K}) \cap cl_\cap(Ч). \end{aligned}$$

(11)

$$\begin{aligned} \square(int_\cap(\mathbb{K})) =& \square(\{\langle \xi, \varepsilon_{\mathbb{K}}, \epsilon_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\}) = \{\langle \xi, \varepsilon_{\mathbb{K}}, 1 - \varepsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\} \\ \subseteq& \{\langle \xi, \varepsilon_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \,|\xi \in \Xi\} = int_\cap(\square(\mathbb{K})). \end{aligned}$$

□

Example 2.7. *Let* $\Xi = \{\xi_1, \xi_2\}$. *Then,*

(1) *For* $\mathbb{K} \in \mathcal{P}(\sharp)$, $\mathbb{K} = \{\langle \xi_1, 0.1, 0.2, 0.3\rangle\ \langle \xi_2, 0, 0.3, 0.5\rangle \mid \xi_1, \xi_2 \in \Xi\}$,
$\mathrm{int}_\cap(\mathbb{K}) = \{\langle \xi, 0, 0.3, 0.3\rangle \mid \xi \in \Xi\}$, $\quad \Box(\mathrm{int}_\cap(\mathbb{K})) = \{\langle \xi, 0, 0.7, 0.3\rangle \mid \xi \in \Xi\}$,
$\Box(\mathbb{K}) = \{\langle \xi_1, 0.1, 0.6, 0.3\rangle\ \langle \xi_2, 0, 0.5, 0.5\rangle \mid \xi_1, \xi_2 \in \Xi\}$,
$\mathrm{int}_\cap(\Box(\mathbb{K})) = \{\langle \xi, 0, 0.6, 0.3\rangle \mid \xi \in \Xi\} \neq \Box(\mathrm{int}_\cap(\mathbb{K}))$.
(2) *For* $\mathbb{K} \in \mathcal{P}(\sharp)$, $\mathbb{K} = \{\langle \xi_1, 0.2, 0.3, 0.5\rangle\ \langle \xi_2, 0.1, 0.6, 0.3\rangle \mid \xi_1, \xi_2 \in \Xi\}$,
$\mathrm{cl}_\cap(\mathbb{K}) = \{\langle \xi, 0.2, 0.3, 0.3\rangle \mid \xi \in \Xi\}$, $\quad \Diamond(\mathrm{cl}_\cap(\mathbb{K})) = \{\langle \xi, 0.4, 0.3, 0.3\rangle \mid \xi \in \Xi\}$,
$\Diamond(\mathbb{K}) = \{\langle \xi_1, 0.2, 0.3, 0.5\rangle\ \langle \xi_2, 0.1, 0.6, 0.3\rangle \mid \xi_1, \xi_2 \in \Xi\}$,
$\mathrm{cl}_\cap(\Diamond(\mathbb{K})) = \{\langle \xi, 0.2, 0.3, 0.3\rangle \mid \xi \in \Xi\} \neq \Diamond(\mathrm{cl}_\cap(\mathbb{B}))$.

Note: If we take $\sigma_{\mathbb{K}}(\xi) = 0$ for all $\xi \in \Xi$, then we obtain the equality in the given axioms (10, 11, 12, 13) as in Atanassov [8].

Corollary 2.8. *For every* $\mathbb{K} \in \mathcal{P}(\sharp)$, *we have the following:*

(1) $\Box(\mathrm{cl}_\cap(\Box(\mathbb{K}))) = \exists\Diamond(\mathrm{int}_\cap(\Diamond(\exists\mathbb{K}))) = \{\langle \xi, \epsilon_{\mathbb{K}}, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(2) $\Diamond(\mathrm{cl}_\cap(\Box(\mathbb{K}))) = \exists\Box(\mathrm{int}_\cap(\Diamond(\exists\mathbb{K}))) \supseteq \{\langle \xi, \epsilon_{\mathbb{K}}, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(3) $\Box(\mathrm{cl}_\cap(\Diamond(\mathbb{K}))) = \exists\Diamond(\mathrm{int}_\cap(\Box(\exists\mathbb{K}))) \supseteq \{\langle \xi, 1 - \aleph_{\mathbb{K}} - \varkappa_{\mathbb{K}}, \aleph_{\mathbb{K}} + \varkappa_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(4) $\Diamond(\mathrm{cl}_\cap(\Diamond(\mathbb{K}))) = \exists\Box(\mathrm{int}_\cap(\Box(\exists\mathbb{K}))) = \{\langle \xi, 1 - \aleph_{\mathbb{K}} - \kappa_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(5) $\Box(\mathrm{int}_\cap(\Box(\mathbb{K}))) = \exists\Diamond(\mathrm{cl}_\cap(\Diamond(\exists\mathbb{K}))) = \{\langle \xi, \varepsilon_{\mathbb{K}}, 1 - \varepsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(6) $\Diamond(\mathrm{int}_\cap(\Box(\mathbb{K}))) = \exists\Box(\mathrm{cl}_\cap(\Diamond(\exists\mathbb{K}))) \subseteq \{\langle \xi, \varepsilon_{\mathbb{K}} + \varkappa_{\mathbb{K}} - \kappa_{\mathbb{K}}, 1 - \varepsilon_{\mathbb{K}} - \varkappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(7) $\Box(\mathrm{int}_\cap(\Diamond(\mathbb{K}))) = \exists\Diamond(\mathrm{cl}_\cap(\Box(\exists\mathbb{K}))) \subseteq \{\langle \xi, 1 - \vartheta_{\mathbb{K}} - \kappa_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(8) $\Diamond(\mathrm{int}_\cap(\Diamond(\mathbb{K}))) = \exists\Box(\mathrm{cl}_\cap(\Box(\exists\mathbb{K}))) = \{\langle \xi, 1 - \vartheta_{\mathbb{K}} - \kappa_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(9) $\Box(\mathrm{cl}_\cap(\Box(\exists\mathbb{K}))) = \exists\Diamond(\mathrm{int}_\cap(\Diamond(\mathbb{K}))) = \{\langle \xi, \vartheta_{\mathbb{K}}, 1 - \vartheta_{\mathbb{K}}\kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(10) $\Diamond(\mathrm{cl}_\cap(\Box(\exists\mathbb{K}))) = \exists\Box(\mathrm{int}_\cap(\Diamond(\mathbb{K}))) \supseteq \{\langle \xi, \vartheta_{\mathbb{K}}, 1 - \vartheta_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(11) $\Box(\mathrm{cl}_\cap(\Diamond(\exists\mathbb{K}))) = \exists\Diamond(\mathrm{int}_\cap(\Box(\mathbb{K}))) \supseteq \{\langle \xi, 1 - \varepsilon_{\mathbb{K}} - \varkappa_{\mathbb{K}}, \varepsilon_{\mathbb{K}} + \varkappa_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(12) $\Diamond(\mathrm{cl}_\cap(\Diamond(\exists\mathbb{K}))) = \exists\Box(\mathrm{int}_\cap(\Box(\mathbb{K}))) = \{\langle \xi, 1 - \varepsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \varepsilon_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(13) $\Box(\mathrm{int}_\cap(\Box(\exists\mathbb{K}))) = \exists\Diamond(\mathrm{cl}_\cap(\Diamond(\mathbb{K}))) = \{\langle \xi, \aleph_{\mathbb{K}}, 1 - \aleph_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(14) $\Diamond(\mathrm{int}_\cap(\Box(\exists\mathbb{K}))) = \exists\Box(\mathrm{cl}_\cap(\Diamond(\mathbb{K}))) \subseteq \{\langle \xi, \aleph_{\mathbb{K}} + \varkappa_{\mathbb{K}} - \kappa_{\mathbb{K}}, 1 - \aleph_{\mathbb{K}} - \varkappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(15) $\Box(\mathrm{int}_\cap(\Diamond(\exists\mathbb{K}))) = \exists\Diamond(\mathrm{cl}_\cap(\Box(\mathbb{K}))) \subseteq \{\langle \xi, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \epsilon_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$;
(16) $\Diamond(\mathrm{int}_\cap(\Diamond(\exists\mathbb{K}))) = \exists\Box(\mathrm{cl}_\cap(\Box(\mathbb{K}))) = \{\langle \xi, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \epsilon_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$.

Example 2.9. *Let* $\Xi = \{\xi_1, \xi_2\}$. *Then,*

(1) *For* $\mathbb{K} \in \mathcal{P}(\sharp)$, $\mathbb{K} = \{(\xi_1, 0.2, 0.2, 0.1),\ (\xi_2, 0.4, 0.3, 0.1) \mid \xi_1, \xi_2 \in \Xi\}$,
$\Box(\mathbb{K}) = \{\langle \xi_1, 0.2, 0.7, 0.1\rangle\ \langle \xi_2, 0.4, 0.5, 0.1\rangle \mid \xi_1, \xi_2 \in \Xi\}$,
$\mathrm{int}_\cap(\Box(\mathbb{K})) = \{\langle \xi, 0.2, 0.7, 0.1\rangle \mid \xi \in \Xi\}$,
$\Diamond(\mathrm{int}_\cap(\Box(\mathbb{K}))) = \{\langle \xi, 0.2, 0.7, 0.1\rangle \mid \xi \in \Xi\}$, *while*
$\{\langle \xi, \epsilon_{\mathbb{K}}, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\} = \{\langle \xi, 0.4, 0.5, 0.1\rangle \mid \xi \in \Xi\}$, *and then*
$\Diamond(\mathrm{int}_\cap(\Box(\mathbb{K}))) \neq \{\langle \xi, \epsilon_{\mathbb{K}}, 1 - \epsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$.
(2) *For* $\mathbb{K} \in \mathcal{P}(\sharp)$, $\mathbb{K} = \{(\xi_1, 0.3, 0.4, 0.2),\ (\xi_2, 0.1, 0.1, 0.6) \mid \xi_1, \xi_2 \in \Xi\}$,
$\Diamond\mathbb{K} = \{(\xi_1, 0.4, 0.4, 0.2)\ (\xi_2, 0.3, 0.1, 0.6) \mid \xi_1, \xi_2 \in \Xi\}$,
$\mathrm{cl}_\cap(\Diamond(\mathbb{K})) = \{\langle \xi, 0.4, 0.1, 0.2\rangle \mid \xi \in \Xi\}$, $\Box(\mathrm{cl}_\cap(\Diamond(\mathbb{K}))) = \{\langle \xi, 0.4, 0.4, 0.2\rangle \mid \xi \in \Xi\}$,
$\mathrm{int}_\cap(\Diamond(\mathbb{K})) = \{\langle \xi, 0.3, 0.4, 0.2\rangle \mid \xi \in \Xi\}$, $\Box(\mathrm{int}_\cap(\Diamond(\mathbb{K}))) = \{\langle \xi, 0.3, 0.5, 0.2\rangle \mid \xi \in \Xi\}$,
$\{\langle \xi, 1 - \aleph_{\mathbb{K}} - \kappa_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\} = \{\langle \xi, 0.7, 0.1, 0.2\rangle \mid \xi \in \Xi\}$,
$\{\langle \xi, 1 - \vartheta_{\mathbb{K}} - \kappa_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\} = \{\langle \xi, 0.4, 0.4, 0.2\rangle \mid \xi \in \Xi\}$,
$\Box(\mathrm{cl}_\cap(\Diamond(\mathbb{K}))) \neq \{\langle \xi, 1 - \aleph_{\mathbb{K}} - \kappa_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}}\rangle \mid \xi \in \Xi\}$,

$\Box(\text{int}_{\cap}(\Diamond(\mathbb{K}))) \neq \{\langle \xi, 1-\vartheta_{\mathbb{K}}-\kappa_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}} \rangle | \xi \in \Xi\}$.

Note: If we take $\sigma_{\mathbb{K}}(\xi) = 0$, for all $\xi \in \Xi$, then we obtain the equality in the given axioms in (2, 3, 6, 7, 10, 11, 14, 15) as in Atanassov [8].

Theorem 2.10. (1) $\langle \mathcal{P}(\sharp), \text{cl}_{\Cup}, \Cup, \Diamond \rangle$ *is a* cl-*PFMT.*
(2) $\langle \mathcal{P}(\sharp), \text{int}_{\Cap}, \Cap, \Box \rangle$ *is an* int- *PFMT.*

3. Picture fuzzy topological multifunctions

The map $\mathbb{F} : \Xi \looparrowright \Upsilon$ is called a PFTM for any $(\xi, \zeta) \in \Xi \times \Upsilon$ iff $\mathbb{F}(\xi) \in PFS(\Upsilon)$ for each $\xi \in \Xi$. The degree of membership of ζ in $\mathbb{F}(\xi)$ is denoted by $\mathbb{F}(\xi)(\zeta) = \Psi_{\mathbb{F}}(\xi, \zeta)$. The domain of $\mathbb{F}$, denoted by $D(\mathbb{F})$, and the range of $\mathbb{F}$, denoted by $R(\mathbb{F})$, are defined by the following: for any $\xi \in \Xi$ and $\zeta \in \Upsilon$, $D(\mathbb{F})(\xi) = \bigcup_{\zeta \in \Upsilon} \Psi_{\mathbb{F}}(\xi, \zeta)$ and $R(\mathbb{F})(\zeta) = \bigcup_{\xi \in \Xi} \Psi_{\mathbb{F}}(\xi, \zeta)$. $\mathbb{F}$ is called crisp iff $\Psi_{\mathbb{F}}(\xi, \zeta) = \langle 1, 0, 0 \rangle$ $\forall \xi \in \Xi$ and $\zeta \in \Upsilon$. $\mathbb{F}$ is called Normalized iff $\forall \xi \in \Xi$, there exists $\zeta_0 \in \Upsilon$ such that $\Psi_{\mathbb{F}}(\xi, \zeta_0) = \langle 1, 0, 0 \rangle$. $\mathbb{F}$ is called surjective iff $R(\mathbb{F})(\zeta) = \langle 1, 0, 0 \rangle$ $\forall \zeta \in \Upsilon$. The inverse of $\mathbb{F}$, denoted by $\mathbb{F}^{-} : \Upsilon \to \Xi$, is a PFM defined by $\mathbb{F}^{-}(\zeta)(\xi) = \mathbb{F}(\xi)(\zeta) = \Psi_{\mathbb{F}}(\xi, \zeta)$. One can easily verify that $D(\mathbb{F}^{-}) = R(\mathbb{F})$ and $D(\mathbb{F}) = R(\mathbb{F}^{-})$. The image $\mathbb{F}(\mathbb{K})$ of $\mathbb{K} \in PFS(\Xi)$, the lower inverse $\mathbb{F}^{l}(\text{Ч})$ of $\text{Ч} \in PFS(\Upsilon)$, and the upper inverse $\mathbb{F}^{u}(\text{Ч})$ of $\text{Ч} \in PFS(\Upsilon)$ are defined, respectively, as follows:

$$\mathbb{F}(\mathbb{K})(\zeta) = \bigcup_{\xi \in \Xi} [\Psi_{\mathbb{F}}(\xi, \zeta) \cap \mathbb{K}(\xi)],$$

$$\mathbb{F}^{l}(\text{Ч})(\xi) = \bigcup_{\zeta \in \Upsilon} [\Psi_{\mathbb{F}}(\xi, \zeta) \cap \text{Ч}(\zeta)],$$

$$\mathbb{F}^{u}(\text{Ч})(\xi) = \bigcap_{\zeta \in \Upsilon} [\exists \Psi_{\mathbb{F}}(\xi, \zeta) \cup \text{Ч}(\zeta)].$$

Definition 3.1. *A picture fuzzy topology on Ξ is a map $\tau : PFS(\Xi) \to I^3$ defined by $\tau(\mathbb{K}) = \langle \omega_\tau(\mathbb{K}), \varpi_\tau(\mathbb{K}), \sigma_\tau(\mathbb{K}) \rangle$ on Ξ, which satisfies the following properties:*

(1) $\tau(\flat) = \tau(\sharp) = \langle 1, 0, 0 \rangle$,

(2) $\tau(\mathbb{K}_1 \cap \mathbb{K}_2) \geq \tau(\mathbb{K}_1) \wedge \tau(\mathbb{K}_2)$, *for each* $\mathbb{K}_1, \mathbb{K}_2 \in PFS(\Xi)$,

(3) $\tau(\bigcup_{i \in \Gamma} \mathbb{K}_i) \geq \bigwedge_{i \in \Gamma} \tau(\mathbb{K}_i)$, *for each* $\mathbb{K}_i \in PFS(\Xi)$, $i \in \Gamma$.

The pair (Ξ, τ) is called a picture fuzzy topological space in Šostak's sense. For any $\mathbb{K} \in PFS(\Xi)$, the number $\omega_\tau(\mathbb{K})$ is called the openness degree, $\varpi_\tau(\mathbb{K})$ is called the non openness degree, while $\sigma_\tau(\mathbb{K})$ is called the neutral degree. For $\mathbb{K} \in PFS(\Xi)$, $cl(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = \bigcap \{\text{Ч} \in PFS(\Xi) : \mathbb{K} \subseteq \text{Ч}, \tau(\exists \text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle\}$, $int(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = \bigcup \{\text{Ч} \in PFS(\Xi) : \mathbb{K} \supseteq \text{Ч}, \tau(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle\}$.

Definition 3.2. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be PFTM, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$. Then, $\mathbb{F}$ is called:*

(1) *PF^uS-continuous at a point $\xi_{\langle m,n,t \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle m,n,t \rangle} \in \mathbb{F}^u(\text{Ч})$ for each $\text{Ч} \in PFS(\Upsilon), \tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, and $\xi_{\langle m,n,t \rangle} \in \mathbb{K}$ such that $\mathbb{K} \cap D(\mathbb{F}) \subseteq \mathbb{F}^u(\text{Ч})$;*

(2) *PF^lS-continuous at a point $\xi_{\langle m,n,t \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle m,n,t \rangle} \in \mathbb{F}^l(\text{Ч})$ for each $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, and $\xi_{\langle m,n,t \rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^l(\text{Ч})$;*

(3) *PF^uA-continuous at a point $\xi_{\langle m,n,t \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle m,n,t \rangle} \in \mathbb{F}^u(\text{Ч})$ for each $\text{Ч} \in PFS(\Upsilon), \tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, and $\xi_{\langle m,n,t \rangle} \in \mathbb{K}$ such that $\mathbb{K} \cap D(\mathbb{F}) \subseteq \mathbb{F}^u(int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle))$;*

(4) PF^lA-*continuous at a point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ *iff* $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч)$ *for each* $Ч \in PFS(\Upsilon)$, $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$ *there exists* $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, *and* $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ *such that* $\mathbb{K} \subseteq \mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$;

(5) PF^uS (PF^lS)-*continuous iff it is* PF^uS (PF^lS)-*continuous at every point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$;

(6) PF^uA (PF^lA)-*continuous iff it is* PF^uA (PF^lA)-*continuous at every point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$.

Remark 3.3. (1) *If* $\mathbb{F}$ *is normalized, then* $\mathbb{F}$ *is* PF^uS-*continuous at a point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ *iff* $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(Ч)$ *for each* $Ч \in PFS(\Upsilon)$, $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$ *there exists* $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, *and* $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ *such that* $\mathbb{K} \subseteq \mathbb{F}^u(Ч)$. *Equivalently,* $\tau(\mathbb{F}^u(Ч)) \geq \tau^*(Ч)$.

(2) *If* $\mathbb{F}$ *is normalized, then* $\mathbb{F}$ *is* PF^uS-*continuous at a point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ *iff* $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(Ч)$ *for each* $Ч \in PFS(\Upsilon)$, $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, *there exists* $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, *and* $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ *such that* $\mathbb{K} \subseteq \mathbb{F}^u(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$.

(3) PF^uS (*resp.* PF^lS)-*continuity* $\Longrightarrow$ PF^uA (*resp.* PF^lA)-*continuity.*

Theorem 3.4. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a PFTM. Then, for* $Ч \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, *and* $\vartheta \in I_1$, *the following are equivalent:*

(1) $\mathbb{F}$ is PF^lA-continuous;

(2) $\mathbb{F}^l(Ч) \subseteq int(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$, if $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$;

(3) $\tau(\mathbb{F}^l(Ч)) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, if $Ч = int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)$;

(4) $\tau(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))) \geq \tau^*(Ч)$.

Proof. (1) $\Longrightarrow$ (2) Let $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$, $Ч \in PFS(\Upsilon), \tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч)$. Then, there exists $\mathbb{K} \in PFS(\Xi)$, $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$. Thus,

$$\xi_{\langle m,n,t\rangle} \in \mathbb{K} \subseteq \mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)),$$

and hence

$$\begin{aligned} \mathbb{F}^l(Ч) &\subseteq \mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)) \\ &\subseteq int(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle). \end{aligned}$$

(2) $\Longrightarrow$ (3) Let $Ч = int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$. Then by (2),

$$\mathbb{F}^l(Ч) \subseteq int(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle) = int(\mathbb{F}^l(Ч), \langle \varsigma, \varkappa, \vartheta\rangle).$$

Thus, $\tau(\mathbb{F}^l(Ч)) \geq \langle \varsigma, \varkappa, \vartheta\rangle$.

(3) $\Longrightarrow$ (4) Suppose that there exist $Ч \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$ such that

$$\tau(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))) < \langle \varsigma, \varkappa, \vartheta\rangle \leq \tau^*(Ч).$$

Since $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$ and $int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) = int(cl(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)$.
Then, by (3), we have $\tau(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))) \geq \langle \varsigma, \varkappa, \vartheta\rangle$. It is a contradiction.
Thus, $\tau(\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))) \geq \tau^*(Ч)$.

(4) $\Longrightarrow$ (1) Let $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$, $Ч \in PFS(\Upsilon), \tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч)$.
Then, by (4), we have $\mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)) = \mathbb{K}$ with $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$.
Since $Ч \subseteq int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)$, then

$$\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч) \subseteq \mathbb{F}^l(int(cl(Ч, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)) = \mathbb{K} \text{ and (1) follows.}$$

□

Theorem 3.5. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a PFTM. Then, for* $\text{Ч} \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, *and* $\vartheta \in I_1$, *the following are equivalent:*

(1) $\mathbb{F}$ is PF^lA-continuous;
(2) $\mathbb{F}^u(\text{Ч}) \supseteq cl(\mathbb{F}^u(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$, if $\tau^*(\dashv \text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$;
(3) $\tau(\dashv(\mathbb{F}^u(\text{Ч}))) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, if $\text{Ч} = cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$.

Proof. (1) $\Longrightarrow$ (2) Let $\mathbb{F}$ be PF^lA-continuous and $\text{Ч} \in PFS(\Upsilon)$ with $\tau^*(\dashv \text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$. Then, by Theorem 3.4(2),

$$
\begin{aligned}
\mathbb{F}^l(\dashv \text{Ч}) &\subseteq int(\mathbb{F}^l(int(cl(\dashv \text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \\
&= int(\mathbb{F}^l(int(\dashv(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \\
&= int(\mathbb{F}^l(\dashv(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle))), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle).
\end{aligned}
$$

Since

$$
\begin{aligned}
\dashv(\mathbb{F}^u(\text{Ч})) &= \mathbb{F}^l(\dashv \text{Ч}) \\
&\subseteq int(\mathbb{F}^l(\dashv(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle))), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \\
&= int(\dashv\mathbb{F}^u(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \\
&= \dashv(cl(\mathbb{F}^u(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)).
\end{aligned}
$$

Thus, we obtain $\mathbb{F}^u(\text{Ч}) \supseteq cl(\mathbb{F}^u(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$.

(2) $\Longrightarrow$ (3) Let $\text{Ч} = cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$. Then, by (2),

$$
\begin{aligned}
\mathbb{F}^u(\text{Ч}) &\supseteq cl(\mathbb{F}^u(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \\
&= cl(\mathbb{F}^u(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)).
\end{aligned}
$$

Thus, $\tau(\dashv(\mathbb{F}^u(\text{Ч}))) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

(3) $\Longrightarrow$ (1) Let $\text{Ч} = int(cl(\text{Ч}, \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle) \in PFS(\Upsilon)$. Then, $\dashv \text{Ч} = cl(int(\dashv \text{Ч}, \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$, and hence, by (3), $\tau(\dashv(\mathbb{F}^u(\dashv \text{Ч}))) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, that is, $\tau(\mathbb{F}^l(\text{Ч})) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

Then, by Theorem 3.4(3), $\mathbb{F}$ is PF^lA-continuous. □

Theorem 3.6. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a PFTM. Then, for* $\text{Ч} \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, *and* $\vartheta \in I_1$, *the following are equivalent:*

(1) $\mathbb{F}$ is PF^uA-continuous;
(2) $\mathbb{F}^u(\text{Ч}) \subseteq int(\mathbb{F}^u(int(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$, if $\tau^*(\text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$;
(3) $\tau(\mathbb{F}^u(\text{Ч})) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, if $\text{Ч} = int(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$;
(4) $\tau(\mathbb{F}^u int(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))) \geq \tau^*(\text{Ч})$;
(5) $\mathbb{F}^l(\text{Ч}) \supseteq cl(\mathbb{F}^l(cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$, if $\tau^*(\dashv \text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$;
(6) $\tau(\dashv(\mathbb{F}^l(\text{Ч}))) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, if $\text{Ч} = cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$.

The following example shows that generally a PF^lA-continuous and PF^uA-continuous multifunction need not be either PF^lS continuous or PF^uS-continuous.

Example 3.7. *Let* $\Xi = \{\xi_1, \xi_2\}$ *,* $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3\}$*, and a PFTM* $\mathbb{F} : \Xi \hookrightarrow \Upsilon$ *be defind by* $\Psi_{\mathbb{F}}(\xi_1,\zeta_1) = \langle 1,0,0\rangle$*,* $\Psi_{\mathbb{F}}(\xi_1, \zeta_2) = \langle 0.1, 0.7, 0.1\rangle$*,* $\Psi_{\mathbb{F}}(\xi_1, \zeta_3) = \langle 0.25, 0.25, 0.5\rangle$*,* $\Psi_{\mathbb{F}}(\xi_2, \zeta_1) = \langle 0.33, 0.33, 0.25\rangle$*,* $\Psi_{\mathbb{F}}(\xi_2, \zeta_2) = \langle 0.1, 0.1, 0.1\rangle$*, and* $\Psi_{\mathbb{F}}(\xi_2, \zeta_3) = \langle 1, 0, 0\rangle$ *.*

Define picture fuzzy topologies τ, τ^* *on* Ξ *and* Υ *respectively, for* $\mathbb{K}_1 = \{\langle\xi_1, 0.1, 0.3, 0.3\rangle, \langle\xi_2, 0.1, 0.5, 0.1\rangle\}$*,* $\text{Ч}_1 = \{\langle\zeta, 0.05, 0.33, 0.5\rangle \,|\zeta \in \Upsilon\}$*, and* $\text{Ч}_2 = \{\langle\zeta, 0.1, 0.3, 0.5\rangle \,|\zeta \in \Upsilon\}$ *as follows:*

$$\tau(\mathbb{K}) = \begin{cases} \langle 1, 0, 0\rangle, & \mathbb{K} \in \{\flat, \sharp\}, \\ \langle 0.5, 0.25, 0.25\rangle, & \mathbb{K} = \mathbb{K}_1, \\ \langle 0, 1, 0\rangle, & o.w, \end{cases}$$

$$\tau^*(\text{Ч}) = \begin{cases} \langle 1, 0, 0\rangle, & \text{Ч} \in \{\flat, \sharp\}, \\ \langle 0.33, 0.5, 0.1\rangle, & \text{Ч} = \text{Ч}_1, \\ \langle 0.5, 0.25, 0.25\rangle, & \text{Ч} = \text{Ч}_2, \\ \langle 0, 1, 0\rangle, & o.w. \end{cases}$$

Then, $\mathbb{F}$ *is* PF^uA *(resp.* PF^lA*)-continuous but it is not* PF^uS *(resp.* PF^lS *)-continuous because*

$$\begin{aligned} \mathbb{F}^u(\text{Ч}_1) &\subseteq int(\mathbb{F}^u(int(cl(\text{Ч}_1, \langle 0.33, 0.5, 0.1\rangle), \langle 0.33, 0.5, 0.1\rangle)), \langle 0.33, 0.5, 0.1\rangle), \\ \mathbb{F}^u(\text{Ч}_2) &\subseteq int(\mathbb{F}^u(int(cl(\text{Ч}_2, \langle 0.33, 0.5, 0.1\rangle), \langle 0.33, 0.5, 0.1\rangle)), \langle 0.33, 0.5, 0.1\rangle), \\ \mathbb{F}^l(\text{Ч}_1) &\subseteq int(\mathbb{F}^l(int(cl(\text{Ч}_1, \langle 0.33, 0.5, 0.1\rangle), \langle 0.33, 0.5, 0.1\rangle)), \langle 0.33, 0.5, 0.1\rangle), \\ \mathbb{F}^l(\text{Ч}_2) &\subseteq int(\mathbb{F}^l(int(cl(\text{Ч}_2, \langle 0.33, 0.5, 0.1\rangle), \langle 0.33, 0.5, 0.1\rangle)), \langle 0.33, 0.5, 0.1\rangle), \end{aligned}$$

but

$$\begin{aligned} \tau(\mathbb{F}^u(\text{Ч}_1)) &= \langle 0, 1, 0\rangle \ngeq \tau^*(\text{Ч}_1) = \langle 0.33, 0.5, 0.1\rangle, \\ \tau(\mathbb{F}^l(\text{Ч}_1)) &= \langle 0, 1, 0\rangle \ngeq \tau^*(\text{Ч}_1) = \langle 0.33, 0.5, 0.1\rangle. \end{aligned}$$

Theorem 3.8. *Let* $\mathbb{F} : (\Xi, \tau) \hookrightarrow (\Upsilon, \tau^*)$ *be a PFTM. Then,* $\mathbb{F}$ *is* PF^lA*-continuous iff* $cl(\mathbb{F}^u(\text{Ч}), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle))$ *for any* $\text{Ч} \subseteq cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$*,* $\varsigma \in I_0, \varkappa \in I_1$*, and* $\vartheta \in I_1$*.*

Proof. Let $\mathbb{F}$ be PF^lA-continuous. Then, for each $\text{Ч} \subseteq cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$, $cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) = \mathfrak{C}$ (say), where $\mathfrak{C} = cl(int(\mathfrak{C}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$. By Theorem 3.5(3), $\tau(\dashv(\mathbb{F}^u(\mathfrak{C}))) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, and thus

$$cl(\mathbb{F}^u(\text{Ч}), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq cl(\mathbb{F}^u(\mathfrak{C}), \langle\varsigma, \varkappa, \vartheta\rangle)) = \mathbb{F}^u(cl(I_{\sigma,\sigma^*}(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)).$$

Conversely, if $\text{Ч} = cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, then $\text{Ч} \subseteq cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, and we have $cl(\mathbb{F}^u(\text{Ч}), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)) = \mathbb{F}^u(\text{Ч})$. Thus, $\tau(\dashv(\mathbb{F}^u\ \text{Ч}))) \geq \langle\varsigma, \varkappa, \vartheta\rangle$. Hence, by Theorem 3.5(3), $\mathbb{F}$ is PF^lA-continuous. □

Theorem 3.9. *Let* $\mathbb{F} : (\Xi, \tau) \hookrightarrow (\Upsilon, \tau^*)$ *be a normalized PFTM. Then,* $\mathbb{F}$ *is* PF^uA*-continuous iff* $cl(\mathbb{F}^l(\text{Ч}), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathbb{F}^l(cl(\text{Ч}), \langle\varsigma, \varkappa, \vartheta\rangle))$ *for any* $\text{Ч} \subseteq cl(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$*,* $\varsigma \in I_0, \varkappa \in I_1$*, and* $\vartheta \in I_1$*.*

4. Picture fuzzy weakly and almost weakly continuous multifunctions

In this section, the notion of PFTM is used, and some of its properties and implications are studied. Various types of continuities of any PFTM between PFMTSs are introduced. Some implications are presented for these multifunctions between various PFMTSs.

Definition 4.1. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a PFTM,* $\varsigma \in I_0, \varkappa \in I_1$*, and* $\vartheta \in I_1$*. Then,* $\mathbb{F}$ *is called:*

(1) PF^uW*-continuous at a point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ *iff* $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(Ч)$ *for each* $Ч \in PFS(\Upsilon)$, $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*, and there exists* $\mathbb{K} \in PFS(\Xi)$*,* $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*, and* $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ *such that* $\mathbb{K} \cap D(\mathbb{F}) \subseteq \mathbb{F}^u(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle))$;

(2) PF^lW*-continuous at a point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ *iff* $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч)$ *for each* $Ч \in PFS(\Upsilon), \tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*, and there exists* $\mathbb{K} \in PFS(\Xi)$*,* $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*, and* $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ *such that* $\mathbb{K} \subseteq \mathbb{F}^l(cl(B, \langle \varsigma, \varkappa, \vartheta \rangle))$;

(3) PF^uW *(*PF^lW*) continuous iff it is* PF^uW *(*PF^lW*)-continuous at every point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$.

Remark 4.2. (1) *If* $\mathbb{F}$ *is normalized, then* $\mathbb{F}$ *is* PF^uW*-continuous at a point* $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ *iff* $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(Ч)$ *for each* $Ч \in PFS(\Upsilon), \tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ *there exists* $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*, and* $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ *such that* $\mathbb{K} \subseteq \mathbb{F}^u(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle))$.

(2) PF^uA *(resp.* PF^lA*)-continuity* $\Longrightarrow$ PF^uW *(resp.* PF^lW*)-continuity.*

Theorem 4.3. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a PFTM. Then,* $\mathbb{F}$ *is* PF^lW*-continuous iff* $\mathbb{F}^l(Ч) \subseteq int(\mathbb{F}^l(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)), \langle \varsigma, \varkappa, \vartheta \rangle)$ *for any* $Ч \in PFS(\Upsilon)$ *with* $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*,* $\varsigma \in I_0, \varkappa \in I_1$*, and* $\vartheta \in I_1$.

Proof. Let $\mathbb{F}$ be PF^lW-continuous and $Ч \in PFS(\Upsilon)$ with $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$. Then, if $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч)$, then there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\xi_{\langle m,n,t\rangle} \in \mathbb{K} \subseteq \mathbb{F}^l(cl(Ч \langle \varsigma, \varkappa, \vartheta \rangle))$, and hence, $\mathbb{K} \subseteq int(\mathbb{F}^l(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)), \langle \varsigma, \varkappa, \vartheta \rangle)$. Thus, $\mathbb{F}^l(Ч) \subseteq int(\mathbb{F}^l(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)), \langle \varsigma, \varkappa, \vartheta \rangle)$. Conversely, let $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$, $Ч \in PFS(\Upsilon)$, $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч)$. Then, $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(Ч) \subseteq int(\mathbb{F}^l(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)), \langle \varsigma, \varkappa, \vartheta \rangle) = \mathbb{K}$ (say). Thus, $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ and $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ such that

$$\mathbb{K} = int(\mathbb{F}^l(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)), \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \mathbb{F}^l(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)),$$

and then $\mathbb{F}$ is PF^lW-continuous. □

Theorem 4.4. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a normalized PFTM. Then,* $\mathbb{F}$ *is* PF^uW*-continuous iff* $\mathbb{F}^u(Ч) \subseteq int(\mathbb{F}^u(cl(Ч, \langle \varsigma, \varkappa, \vartheta \rangle)), \langle \varsigma, \varkappa, \vartheta \rangle)$ *for any* $Ч \in PFS(\Upsilon)$ *with* $\tau^*(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$*,* $\varsigma \in I_0, \varkappa \in I_1$*, and* $\vartheta \in I_1$.

The following example shows that generally a PF^uW-continuous and PF^lW-continuous multifunction need not be either PF^uA-continuous or PF^lA-continuous.

Example 4.5. *Let* $\Xi = \{\xi_1, \xi_2\}$ *and* $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3\}$*. A PFM* $\mathbb{F} : \xi \looparrowright \Upsilon$ *is defind by* $\Psi_{\mathbb{F}}(\xi_1, \zeta_1) = \langle 0.3, 0.1, 0.5 \rangle$, $\Psi_{\mathbb{F}}(\xi_1, \zeta_2) = \langle 1, 0, 0 \rangle$, $\Psi_{\mathbb{F}}(\xi_1, \zeta_3) = \langle 0.1, 0.4, 0.2 \rangle$, $\Psi_{\mathbb{F}}(\xi_2, \zeta_1) = \langle 1, 0, 0 \rangle$, $\Psi_{\mathbb{F}}(\xi_2, \zeta 2) = \langle 0.2, 0.4, 0.3 \rangle$*, and* $\Psi_{\mathbb{F}}(\xi_2, \zeta_3) = \langle 0.2, 0.1, 0.7 \rangle$*. Define picture fuzzy topologies* τ*,* τ^* *on* Ξ *and* Υ*, respectively, for* $\mathbb{K}_1 = \{\langle \xi, 0.3, 0.3, 0.1 \rangle, \xi \in \Xi\}$, $Ч_1 = \{\langle \Upsilon, 0.2, 0.5, 0.3 \rangle, \zeta \in \Upsilon\}$*, as follows:*

$$\tau(\mathbb{K}) = \begin{cases} \langle 1, 0, 0 \rangle, & \mathbb{K} \in \{\flat, \sharp\}, \\ \langle 0.33, 0.3, 0.35 \rangle, & \mathbb{K} = \mathbb{K}_1, \\ \langle 0, 1, 0 \rangle, & o.w, \end{cases}$$

$$\tau^*(\text{Ч}) = \begin{cases} \langle 1,0,0\rangle, & \text{Ч} \in \{\flat, \sharp\}, \\ \langle 0.33, 0.33, 0.33\rangle, & \text{Ч} = \text{Ч}_1, \\ \langle 0,1,0\rangle, & o.w. \end{cases}$$

Then, $\mathbb{F}$ is PF^uW (resp. PF^lW) continuous but it is not PF^uA (resp. PF^lA) continuous because

$$\mathbb{F}^u(\text{Ч}_1) = \langle 0.2, 0.5, 0\rangle \subseteq int(\mathbb{F}^u(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle) = \langle 0.5, 0.3, 0\rangle,$$
$$\mathbb{F}^l(\text{Ч}_1) = \langle 0.2, 0.5, 0\rangle \subseteq int(\mathbb{F}^l(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle) = \langle 0.3, 0.3, 0\rangle,$$

but

$$\mathbb{F}^u(\text{Ч}_1) = \langle 0.2, 0.5, 0\rangle \nsubseteq int(\mathbb{F}^u(int(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle) = \langle 0,1,0\rangle,$$
$$\mathbb{F}^l(\text{Ч}_1) = \langle 0.2, 0.5, 0\rangle \nsubseteq int(\mathbb{F}^l(int(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle) = \langle 0,1,0\rangle.$$

Theorem 4.6. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be a PF^lW-continuous multifunction. Then, $\mathbb{F}^l(\text{Ч}) \subseteq int(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$ for any $\text{Ч} = int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, and $\vartheta \in I_1$.*

Proof. Let $\mathbb{F}$ be PF^lW-continuous and $\text{Ч} = int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$. Then, if $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(\text{Ч}) \subseteq \mathbb{F}^l(int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$, then there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that

$$\mathbb{K} \subseteq \mathbb{F}^l(cl(int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)).$$

Thus, $\mathbb{K} \subseteq int(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$, and hence, $\mathbb{F}^l(\text{Ч}) \subseteq int(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$. □

Theorem 4.7. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be a normalized PF^uW-continuous multifunction. Then, $\mathbb{F}^u(\text{Ч}) \subseteq int(\mathbb{F}^u(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$ for any $\text{Ч} = int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, and $\vartheta \in I_1$.*

Theorem 4.8. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be a PFTM, $\mathbb{F}$ be normalized PF^uW-continuous, and for any $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, if $\mathbb{F}(\mathbb{K}) \subseteq int(cl(\mathbb{F}(\mathbb{K}), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)$, then $\mathbb{F}$ is PF^uA-continuous.*

Proof. Let $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$, $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(\text{Ч})$. Then, there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^u(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle))$; thus,

$$F(\mathbb{K}) \subseteq \mathbb{F}(\mathbb{F}^u(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle))) \subseteq cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle).$$

If $\mathbb{F}(\mathbb{K}) \subseteq int(cl(\mathbb{F}(\mathbb{K}), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)$, then

$$\mathbb{F}(\mathbb{K}) \subseteq int(cl(\mathbb{F}(\mathbb{K}), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \subseteq int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle),$$

and hence, $\mathbb{K} \subseteq \mathbb{F}^u(\mathbb{F}(\mathbb{K})) \subseteq \mathbb{F}^u(int(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$. Then, F is PF^uA-continuous. □

Definition 4.9. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be a PFTM, $\varsigma \in I_0, \varkappa \in I_1$, and $\vartheta \in I_1$. Then, $\mathbb{F}$ is called:*

(1) PF^uAW-continuous at a point $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(\text{Ч})$ for each $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{K} \in PFS(\Xi)$, $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\mathbb{K} \cap D(\mathbb{F}) \subseteq cl(\mathbb{F}^u(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$;

(2) PF^lAW-continuous at a point $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(\text{Ч})$ for each $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{K} \in PFS(\Xi)$, $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$;

(3) PF^uAW (PF^lAW)-continuous iff it is PF^uAW (PF^lAW)-continuous at every point $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$.

Remark 4.10. (1) *If $\mathbb{F}$ is normalized, then $\mathbb{F}$ is PF^uAW-continuous at a point $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^u(\text{Ч})$ for each $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{K} \in PFS(\Xi)$, $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq cl(\mathbb{F}^u(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$.*

(2) *PF^uW (resp. PF^lW)-continuity$\Longrightarrow$ PF^uAW (resp. PF^lAW)-continuity.*

Theorem 4.11. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be a PFTM. Then, for $\text{Ч} \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, and $\vartheta \in I_1$, the following are equivalent:*

(1) *$\mathbb{F}$ is PF^lAW-continuous;*

(2) *$\mathbb{F}^l(\text{Ч}) \subseteq int(cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)$, if $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$;*

(3) *$cl(int(\mathbb{F}^u(int(\text{Ч}\langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \subseteq \mathbb{F}^u(\text{Ч})$, if $\tau^*(\text{⅃}\,\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$.*

Proof. (1) $\Longrightarrow$ (2) Let $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$, $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(\text{Ч})$. Then, there exists $\mathbb{K} \in PFS(\Xi), \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$. Thus,

$$\xi_{\langle m,n,t\rangle} \in \mathbb{K} \subseteq int(cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle),$$

and hence,

$$\mathbb{F}^l(\text{Ч}) \subseteq int(cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle).$$

(2) $\Longrightarrow$ (3) Let $\text{Ч} \in PFS(\Upsilon)$ with $\tau^*(\text{⅃}\,\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$. Then, by (2),

$$\begin{aligned} \text{⅃}[\mathbb{F}^u(\text{Ч})] &= \mathbb{F}^l(\text{⅃}\,\text{Ч}) \subseteq int(cl(\mathbb{F}^l(cl(\text{⅃}\,\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \\ &= \text{⅃}[cl(int(\mathbb{F}^u(int(\text{Ч}\langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)]. \end{aligned}$$

Thus,

$$\mathbb{F}^u(\text{Ч}) \supseteq cl(int(\mathbb{F}^u(int(\text{Ч}\langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle).$$

(3) $\Longrightarrow$ (1) Let $\xi_{\langle m,n,t\rangle} \in D(\mathbb{F})$, $\text{Ч} \in PFS(\Upsilon)$, $\tau^*(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\xi_{\langle m,n,t\rangle} \in \mathbb{F}^l(\text{Ч})$. Then, by (3), we have

$$\begin{aligned} &\text{⅃}\left[int(cl(\mathbb{F}^l(cl(\text{⅃}\,\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle)\right] \\ = \;& cl(int(\mathbb{F}^u(int(\text{⅃}\,\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \subseteq \mathbb{F}^u(\text{⅃}\,\text{Ч}) = \text{⅃}\left[\mathbb{F}^l(\text{Ч})\right], \end{aligned}$$

and hence,

$\mathbb{F}^l(\text{Ч}) \subseteq int(cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle) \subseteq cl(\mathbb{F}^l(cl(\text{Ч}, \langle \varsigma, \varkappa, \vartheta\rangle)), \langle \varsigma, \varkappa, \vartheta\rangle)$. Thus, F is PF^lAW-continuous. □

Theorem 4.12. *Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ be a normalized PFTM. Then, for $\text{Ч} \in PFS(\Upsilon)$, $\varsigma \in I_0, \varkappa \in I_1$, and $\vartheta \in I_1$, the following are equivalent:*

(1) $\mathbb{F}$ is PF^uAW-continuous;
(2) $\mathbb{F}^u(\text{Ч}) \subseteq int(cl(\mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, if $\tau^*(\text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$;
(3) $cl(int(\mathbb{F}^l(int(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \subseteq \mathbb{F}^l(\text{Ч})$, if $\tau^*(\dashv \text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

The following example shows that generally a PF^uAW-continuous and PF^lAW-continuous multifunction need not be either PF^uW-continuous or PF^lW-continuous.

Example 4.13. *From Example 4.5, for* $\mathbb{K}_1 = \{\langle\xi, 0.3, 0.1, 0.1\rangle, \xi \in \Xi\}$, $\text{Ч}_1 = \{\langle\zeta, 0.2, 0.5, 0.3\rangle \mid \zeta \in \Upsilon\}$, *define* τ *as follows:*

$$\tau(\mathbb{K}) = \begin{cases} \langle 1, 0, 0\rangle, & \mathbb{K} \in \{\flat, \sharp\}, \\ \langle 0.33, 0.3, 0.35\rangle, & \mathbb{K} = \mathbb{K}_1, \\ \langle 0, 1, 0\rangle, & o.w. \end{cases}$$

Then, $\mathbb{F}$ *is* PF^uAW *(resp.* PF^lAW*)-continuous but it is not* PF^uW *(resp.* PF^lW*)-continuous because* $\mathbb{F}^u(\text{Ч}_1) = \langle 0.2, 0.5, 0\rangle \subseteq int(cl(\mathbb{F}^u(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle) = \sharp$, $\mathbb{F}^l(\text{Ч}_1) = \langle 0.2, 0.5, 0\rangle \subseteq int(cl(\mathbb{F}^l(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle) = \sharp$, *but*

$$\begin{aligned} \mathbb{F}^u(\text{Ч}_1) &= \langle 0.2, 0.5, 0\rangle \not\subseteq int(\mathbb{F}^u(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle) = \langle 0.3, 0.3, 0.1\rangle = \flat, \\ \mathbb{F}^l(\text{Ч}_1) &= \langle 0.2, 0.5, 0\rangle \not\subseteq int(\mathbb{F}^l(cl(\text{Ч}_1, \langle 0.33, 0.33, 0.33\rangle)), \langle 0.33, 0.33, 0.33\rangle) = \langle 0.3, 0.3, 0.1\rangle = \flat. \end{aligned}$$

Theorem 4.14. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a normalized PFTM. If* $\mathbb{F}$ *is* PF^uAW*-continuous and* PF^lA*-continuous, then,* $\mathbb{F}$ *is* PF^uW*-continuous.*

Proof. Let $\text{Ч} \in PFS(\Upsilon)$ with $\tau^*(\text{Ч}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, $\varsigma \in I_0, \varkappa \in I_1$, $\vartheta \in I_1$, and $\mathbb{F}$ be PF^uAW-continuous. Then, by Theorem 4.12(2),

$$\mathbb{F}^u(\text{Ч}) \subseteq int(cl(\mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle).$$

Since

$$cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle) = cl(int(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle),$$

it follows from Theorem 3.5(3) that

$$\tau(\dashv [\mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle))]) \geq \langle\varsigma, \varkappa, \vartheta\rangle, \text{ then } \mathbb{F}^u(\text{Ч}) \subseteq int(\mathbb{F}^u(cl(\text{Ч}, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle).$$

Hence, it follows from Theorem 4.4 that $\mathbb{F}$ is PF^uW-continuous. □

Theorem 4.15. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \tau^*)$ *be a normalized PFTM. If* $\mathbb{F}$ *is* PF^lAW*-continuous and* PF^uA*-continuous, then,* $\mathbb{F}$ *is* PF^lW*-continuous.*

5. Conclusions

In this paper, the definitions of the two standard modal operators □ and ◇, for PFSs and is introduced two PFMTSs were introduced. Two other new PFMTSs were examined, which generated new types of closure and interior operators in the PFTSs, thus differing from those defined initially and aligning with the specific conditions in the PFMTSs. In these new PFMTSs, the equality in some common conditions were not satisfied as stated in [8,10], and thus it needs to be adjusted to either "⊆" or "⊇". The PFMTSs constructed in this paper are described as "feeble", denoted by PFFMTSs. For future work, we aim to develop other PFMTSs and PFFMTSs. Additionally, we will create other operators using integration, differentiation, and several distinct notions within the proposed PFMTSs. The idea of PFMTSs builds the framework to address uncertainty, imprecision, and neutrality in these more complicated systems. PFMTSs could be used in the fields of decision making, artificial intelligence, natural language processing, robotics, and data mining, thereby exploring its ability to enhance decision-making processes, improve algorithmic robustness, and enable nuanced data analyses. The future research on PFMTSs may focus on their application in emerging fields such as quantum computations, dynamical systems, and hybrid decision-making modules.

Author contributions

M. N. Abu_Shugair: Resources, Methodology, Funding, Writing-original draft; A. A. Abdallah: Validation, Formal analysis, Visualization; Malek Alzoubi: Resources, Methodology, Funding, Visualization; S. E. Abbas: Validation, Formal analysis, Reviewing, Investigation the final version; Ismail Ibedou: Reviewing, Investigation the final version, Writing-original draft. The authors all confirmed this published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. A. Razaq, I. Masmali, H. Garg, U. Shuaib, Picture fuzzy topological spaces and associated continuous functions, *AIMS Mathematics*, **7** (2022), 14840–14861. http://dx.doi.org/10.3934/math.2022814
2. M. Abu_Shugair, A. Abdallah, S. Abbas, I. Ibedou, Double fuzzy α-ð-continuous multifunctions, *AIMS Mathematics*, **9** (2024), 16623–16642. http://dx.doi.org/10.3934/math.2024806
3. M. Abu_Shugair, A. Abdallah, S. Abbas, E. El-Sanowsy, I. Ibedou, Double fuzzy ideal multifunctions, *Mathematics*, **12** (2024), 1128. http://dx.doi.org/10.3390/math12081128
4. T. Al-shami, A. Mhemdi, Generalized frame for orthopair fuzzy sets: (m, n)-fuzzy sets and their applications to multi-criteria decision-making methods, *Information*, **14** (2023), 56. http://dx.doi.org/10.3390/info14010056

5. I. Alshammari, P. Mani, C. Ozel, H. Garg, Multiple attribute decision making algorithm via picture fuzzy nano topological spaces, *Symmetry*, **13** (2021), 69. http://dx.doi.org/10.3390/sym13010069
6. S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, *J. Intell. Fuzzy Syst.*, **36** (2019), 2829–2844. http://dx.doi.org/10.3233/JIFS-172009
7. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. http://dx.doi.org/10.1016/S0165-0114(86)80034-3
8. K. Atanassov, Intuitionistic fuzzy modal topological structure, *Mathematics*, **10** (2022), 3313. http://dx.doi.org/10.3390/math10183313
9. K. Atanassov, R. Tsvetkov, New intuitionistic fuzzy operations, operators and topological structures, *Iran. J. Fuzzy Syst.*, **20** (2023), 37–53. http://dx.doi.org/10.22111/IJFS.2023.7629
10. K. Atanassov, N. Angelova, T. Pencheva, On two intuitionistic fuzzy modal topological structures, *Axioms*, **12** (2023), 408. http://dx.doi.org/10.3390/axioms12050408
11. P. Chellamani, D. Ajay, S. Broumi, T. Ligori, An approach to decision-making via picture fuzzy soft graphs, *Granul. Comput.*, **7** (2022), 527–548. http://dx.doi.org/10.1007/s41066-021-00282-2
12. B. Cuong, Picture fuzzy sets, *Journal of Computer Science and Cybernetics*, **30** (2014), 409. http://dx.doi.org/10.15625/1813-9663/30/4/5032
13. M. Fitting, R. Mendelsohn, *First-order modal logic*, Cham: Springer, 2023. http://dx.doi.org/10.1007/978-3-031-40714-7
14. H. Garg, M. Atef, *Cq*-ROFRS: covering *q*-rung orthopair fuzzy rough sets and its application to multi-attribute decision making process, *Complex Intell. Syst.*, **8** (2022), 2349–2370. http://dx.doi.org/10.1007/s40747-021-00622-4
15. R. Joshi, S. Kumar, A novel VIKOR approach based on weighted correlation coefficients and picture fuzzy information for multicriteria decision making, *Granul. Comput.*, **7** (2022), 323–336. http://dx.doi.org/10.1007/s41066-021-00267-1
16. F. Gündoĝdu, C. Kahraman, Spherical fuzzy sets and spherical fuzzy TOPSIS method, *J. Intell. Fuzzy Syst.*, **36** (2019), 337–352. http://dx.doi.org/10.3233/JIFS-181401
17. L. Li, R. Zhang, J. Wang, X. Shang, K. Bai, A novel approach to multi-attribute group decision-making with *q*-rung picture linguistic information, *Symmetry*, **10** (2018), 172. http://dx.doi.org/10.3390/sym10050172
18. M. Luo, Y. Zhang, A new similarity measure between picture fuzzy sets and its application, *Eng. Appl. Artif. Intel.*, **96** (2020), 103956. http://dx.doi.org/10.1016/j.engappai.2020.103956
19. G. Mints, *A short introduction to modal logic*, Chicago: University of Chicago Press, 1992.
20. M. Olgun, M. Ünver, S. Yardmc, Pythagorean fuzzy points and applications in pattern recognition and Pythagorean fuzzy topologies, *Soft Comput.*, **25** (2021), 5225–5232. http://dx.doi.org/10.1007/s00500-020-05522-2
21. T. Senapati, R. Yager, Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision making methods, *Eng. Appl. Artif. Intell.*, **85** (2019), 112–121. http://dx.doi.org/10.1016/j.engappai.2019.05.012

22. G. Wei, Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making, *Informatica*, **28** (2017), 547–564. http://dx.doi.org/10.3233/INF-2017-1150
23. R. Yager, Pythagorean fuzzy subsets, *Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, 2013, 57–61. http://dx.doi.org/10.1109/IFSA-AFIPS.2013.6608375
24. R. Yager, Generalized orthopair fuzzy sets, *IEEE Trans. Fuzzy Syst.*, **25** (2017), 1222–1230. http://dx.doi.org/10.1109/TFUZZ.2016.2604005
25. Y. Yang, C. Liang, S. Ji, T. Liu, Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making, *J. Intell. Fuzzy Syst.*, **29** (2015), 1711–1722. http://dx.doi.org/10.3233/IFS-151648
26. J. Ye, Similarity measures based on the generalized distance of neutrosophic Z-number sets and their multi-attribute decision making method, *Soft Comput.*, **25** (2021), 13975–13985. http://dx.doi.org/10.1007/s00500-021-06199-x
27. A. Yolcu, F. Smarandache, T. Öztürk, Intuitionistic fuzzy hypersoft sets, *Commun. Fac. Sci. Univ.*, **70** (2021), 443–455. http://dx.doi.org/10.31801/cfsuasmas.788329
28. L. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X